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Transport Phenomena in Degenerate Gases and their Bearing on White Dwarfs.

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ABSTRACT.

Methods for the study of transport problems due to Maxwell and Chapman have been extended to the degenerate gases by applying the new statistics due to Fermi and Dirac. In order to apply this to an assembly of electrons and ions, Perisco's method has been adopted using known data for white dwarfs. Calculations are given for the companion of Sirius and ϵ_2-Eridani for different central temperatures.

Recently Kothari studied the transport phenomena in degenerate gases by applying Boltzman's method [1] and considered the case of inverse square law according to the method due to Chapman.[2] In a later paper [3] he indicates the possibility of the application of his result to white dwarfs. Since the problem is of vital interest not only to dwarf stars but even to the ordinary giant stars, which according to Milne [4] possess a degenerate core, in the present investigation we shall study the problem afresh after the more powerful method due to Maxwell and Chapman [5] and in the application to stellar models we shall

[1] Kothari, Phil. Mag., **13**, 361 (1932); see also Uehling and Uhlenbeck. Phys. Rev., **43**, 552 (1933).

[2] Chapman, M. N., **32**, 291 (1922).

[3] Kothari, *ibid*, **93**.

[4] Milne, *ibid*, **91**, 4 (1930).

[5] Maxwell, Collected Works, II; Chapman. Phil. Trans., **216A**, 279.

adopt the more rigid method of Perisco[6] instead of Chapman's approximation of the inverse square law.

In the new statistics the number of molecules per unit vol. is given by the well-known formula

$$n=\frac{m^3}{h^3}\iiint F(\dot{x},\ \dot{y},\ \dot{z})\ d\dot{x}d\dot{y}d\dot{z} \qquad \ldots \quad (1)$$

the distribution function $F=\frac{1}{Ae^{u/kT}\pm 1}$, the symbols A, u, k and T having their usual significance and plus or minus sign is to be taken according as the Fermi-Dirac or Bose-Einstein statistics is followed.

The average value of any property P is

$$\overline{P}=\frac{\iiint FPd\dot{x}d\dot{y}d\dot{z}}{\iiint Fd\dot{x}d\dot{y}d\dot{z}}. \qquad \ldots \quad (2)$$

Hydrodynamical Equation of Continuity.

With our new value of P and following Maxwell's method we obtain the usual hydrodynamical equation of continuity,[7] *viz.*,

$$\frac{dn}{dt}=-\left[\frac{\partial}{\partial x}(n\dot{x}_0)+\frac{\partial}{\partial y}(n\dot{y}_0)+\frac{\partial}{\partial z}(n\dot{z}_0)\right] \qquad \ldots \quad (3)$$

for the steady state, where x_0 etc. are the velocities due to mass-motion and n the number density.

For the gas not in the steady state we have the general equation

$$\frac{d}{dt}(n\overline{P})=-\left[\frac{\partial}{\partial x}(n\overline{\dot{x}P})+\frac{\partial}{\partial y}(n\overline{\dot{y}P})+\frac{\partial}{\partial z}(n\overline{\dot{z}P})\right]$$

$$+\frac{n}{m}\left[X\frac{\partial\overline{P}}{\partial\dot{x}}+Y\frac{\partial\overline{P}}{\partial\dot{y}}+Z\frac{\partial\overline{P}}{\partial\dot{z}}\right]+\triangle P. \qquad \ldots \quad (4)$$

6 Perisco, M. N., 86, 93 (1926).

7 Jeans, Dynamical Theory of Gases (Cambridge), 4th Ed., Chap. IX, p. 231.

The second term in the above expression is due to the external force and the last one due to collision.

By usual methods of transformation we obtain finally

$$n\frac{D\overline{P}}{Dt}=\sum\left[-\frac{\partial}{\partial x}(n\overline{\xi P})+\frac{n}{m}X\frac{\partial \overline{P}}{\partial \dot{x}_0}\right]+\triangle P \quad \ldots \quad (5)$$

where

$$\frac{D}{Dt}=\frac{d}{dt}+\dot{x}_0\frac{\partial}{\partial x}+\dot{y}_0\frac{\partial}{\partial y}+\dot{z}_0\frac{\partial}{\partial z}.$$

$\sum$ denotes the summation with respect to x, y, z ; ξ, η, ζ are the components of the molecular velocity.

In the case of transfer for a single gas $\triangle P=0$ and eqn. (5) reduces to

$$n\frac{D\dot{x}_0}{Dt}=-\left[\frac{\partial}{\partial x}(n\overline{\xi^2})+\frac{\partial}{\partial y}(n\overline{\xi\eta})+\frac{\partial}{\partial z}(n\overline{\xi\zeta})\right]+\frac{n}{m}X \quad \ldots \quad (6)$$

and corresponding equations for y and z, putting P equal to $\dot{x}$, $\dot{y}$, and $\dot{z}$ respectively.

Eliminating X, Y and Z from the equations (5) and (6) we have the final equation

$$n\left[\frac{D\overline{P}}{Dt}-\frac{\partial\overline{P}}{\partial\dot{x}_0}\cdot\frac{D\dot{x}_0}{Dt}-\frac{\partial\overline{P}}{\partial\dot{y}_0}\cdot\frac{D\dot{y}_0}{Dt}-\frac{\partial\overline{P}}{\partial\dot{z}_0}\frac{D\dot{z}_0}{Dt}\right]$$

$$=\sum\left[-\frac{\partial}{\partial x}(n\overline{\xi P})+\frac{\partial\overline{P}}{\partial\dot{x}_0}\left\{\frac{\partial}{\partial x}(n\overline{\xi^2})+\frac{\partial}{\partial y}(n\overline{\xi\eta})+\frac{\partial}{\partial z}(n\overline{\xi\zeta})\right\}\right]+\triangle P \quad \ldots \quad (7)$$

Taking $P=\dot{x}^2$ we have equation (7) in the following form

$$n\frac{D}{Dt}(\overline{\xi^2})=-\frac{\partial}{\partial x}(n\overline{\xi^3})-\frac{\partial}{\partial y}(n\overline{\xi^2\eta})-\frac{\partial}{\partial z}(n\overline{\xi^2\zeta})$$

$$-2n\left(\overline{\xi^2}\frac{\partial\dot{x}_0}{\partial x}+(\overline{\xi\eta}\frac{\partial\dot{x}_0}{\partial y}+\overline{\xi\zeta}\frac{\partial\dot{x}_0}{\partial z}\right)+\triangle\dot{x}^2 \quad \ldots \quad (8)$$

Now

$$\overline{\xi^2}=\overline{\eta^2}=\overline{\zeta^2}=\lambda \text{ and } \overline{\xi\eta}=\overline{\eta\zeta}=\overline{\xi\zeta}=0.$$

Substituting these in equation (7) we have

$$n\left[\frac{D\overline{P}}{Dt}-\frac{\partial \overline{P}}{\partial \dot{x}_0}\frac{D\dot{x}_0}{Dt}-\frac{\partial \overline{P}}{\partial \dot{y}_0}\frac{D\dot{y}_0}{Dt}-\frac{\partial \overline{P}}{\partial \dot{z}_0}\cdot\frac{D\dot{z}_0}{Dt}\right]$$

$$=\sum\left[-\frac{\partial}{\partial x}(n\overline{\xi P})+\frac{\partial \overline{P}}{\partial \dot{x}_0}\frac{\partial}{\partial x}(n\lambda)\right]+\triangle P \qquad \dots \quad (9)$$

and from equation (8) we have

$$n\frac{D\lambda}{Dt}=-2n\lambda\frac{\partial \dot{x}_0}{\partial x}+\triangle\dot{x}^2 \qquad \dots \quad (10)$$

and two other similar equations.

Adding these and remembering that $\triangle(\dot{x}^2+\dot{y}^2+\dot{z}^2)=0$ we have

$$3\frac{D\lambda}{Dt}=-2\lambda\left(\frac{\partial \dot{x}_0}{\partial x}+\frac{\partial \dot{y}_0}{\partial y}+\frac{\partial \dot{z}_0}{\partial z}\right). \qquad \dots \quad (11)$$

On eliminating $\frac{D\lambda}{Dt}$ between equations (10) and (11)

$$n\lambda\left[2\frac{\partial \dot{x}_0}{\partial x}-\tfrac{2}{3}\left(\frac{\partial \dot{x}_0}{\partial x}+\frac{\partial \dot{y}_0}{\partial y}+\frac{\partial \dot{z}_0}{\partial z}\right)\right]=\triangle\dot{x}^2. \qquad \dots \quad (12)$$

Incidentally we may note that from equations (3) and (11) we get

$$\lambda n^{-\frac{2}{3}}=0$$

since $\lambda=\frac{p}{m}$ for the classical as well as for the degenerate case, we have the general law of adiabetic motion

$$p\rho^{-\frac{5}{3}}=\text{Const.} \qquad \dots \quad (13)$$

We now proceed to calculate the average values of $\triangle$P for $P=\dot x\dot y$ and obtain finally

$$\triangle(\dot x\dot y)=n\lambda\left(\frac{\partial \dot y_0}{\partial x}+\frac{\partial \dot x_0}{\partial y}\right). \qquad \dots \quad (14)$$

We next consider the case of $P=\dot x(\dot x^2+\dot y^2+\dot z^2)$

$$\bar P=\dot x_0(\dot x_0{}^2+\dot y_0{}^2+\dot z_0{}^2)+5\dot x_0\lambda$$

$$\overline{\xi P}=(3\dot x_0{}^2+\dot y_0{}^2+\dot z_0{}^2)\lambda+\overline{\xi^2(\xi^2+\eta^2+\zeta^2)}$$

$$\overline{\eta P}=2\dot x_0\dot y_0\lambda,\ \overline{\zeta P}=2\dot x_0\dot z_0\lambda$$

$\overline{\xi^4}=\frac{15}{7}\lambda^2$ for the degenerate case. (It may be noted that for classical case $\overline{\xi^4}=3\lambda^2$)

$$\overline{\xi^2\eta^2}=\overline{\xi^2\zeta^2}=\tfrac{5}{7}\lambda^2(=\lambda^2 \text{ for the classical case})$$

$$\overline{\xi^2(\xi^2+\eta^2+\zeta^2)}=\tfrac{25}{7}\lambda^2(=5\lambda^2 \qquad ,, \qquad ,, \quad).$$

Hence we have

$$\triangle\dot x(\dot x(\dot x^2+\dot y^2+\dot z^2)$$

$$=-2n\dot x_0\lambda\left[2\frac{\partial \dot x_0}{\partial x}-\tfrac{2}{3}\left(\frac{\partial \dot x_0}{\partial x}+\frac{\partial \dot y_0}{\partial y}+\frac{\partial \dot z_0}{\partial z}\right)\right]+2n\lambda\dot y_0\left(\frac{\partial \dot y_0}{\partial x}+\frac{\partial \dot x_0}{\partial y}\right)$$

$$+2n\lambda\dot z_0\left(\frac{\partial \dot x_0}{\partial z}+\frac{\partial \dot z_0}{\partial x}\right)+\frac{15}{7}n\lambda\frac{\partial\lambda}{\partial x}. \qquad \dots \quad (15)$$

For the classical case the last term in the above equation (right-hand side) has for its numerical coefficient 5 instead of 15/7.

Calculation of $\triangle$P.

The dynamics of collision according to the new statistics[8] takes the form

$$\triangle_{12}P=\int\dots\int F_1\left(1\pm\frac{F_1}{Z_1}\right)F'_2\left(1\pm\frac{F'_2}{Z_2}\right)d\dot x_1 d\dot y_1 d\dot z_1 d\dot x_2 d\dot y_2 d\dot z_2 .$$
$$V[P]pdpd\epsilon \qquad \dots \quad (16)$$

[8] Nordheim, P. R. S., 117A, 258 (1927).

Numerical Calculations.

In order to calculate A_1 and A_2 instead of taking the law of inverse square for the point charge as was done by Chapman we considered a distribution of charge according to Debye and Hückel as assumed by Perisco. We then proceeded to find out the potential and evaluate the integral

$$\theta_o(V,p)=\int_o^{\eta_1}\frac{d\eta}{\sqrt{1+\eta^2-\frac{2e_1}{V^2}\cdot\frac{m_1+m_2}{m_1m_2}\phi\left(\frac{p}{\eta}\right)}}$$

by a graphical method for the cases of Sirius B assuming the central temperature to be of the orders 10^7, 10^8 and 10^9 and for ϵ_2-Eridani with central temperature 10^8 assuming that the stellar matter is composed of completely ionised Ca-atoms.[9] The integrations involved in A_1 and A_2 were subsequently carried on graphically. The physical constants for the two white dwarfs are given in Table I[10] and the computed results in Table II.

TABLE I.

Name of the star.	Density $\times 10^{-4}$	Electron number density.	Velocity $\times 10^{-9}$	Probable central temp.
ϵ_2-Eridani	9·8	$2{\cdot}55 \times 10^{28}$	8·174	10^8
Sirius B	5·0	1·303	6·536	10^9
,,	,,	,,	,,	10^8
,,	,,	,,	,,	10^7

[9] Ganguli, Current Science, Dec., 1932, also Jan., 1934.

[10] Jeans, Astronomy and Cosmogony. In calculating n the average mol. wt. was taken to be 2·2.

TABLE II.

Name of the star.	Temp.	$n_e \times 10^{-29}$	Diffusion D.	Viscosity κ.	Conductivity $\vartheta \times 10^{16}$.
e_2 Eridani	10^8	2·55	$3{\cdot}621 \times 10^{-2}$	1·466	2·458
Sirius B	10^9	1·303	1·937	52·05	2·96
,,	10^8	,,	1·071	28·73	1·629
,,	$1{\cdot}37 \times 10^7$	,,	1·121	69·02	3·619

The densities of the two other known white dwarfs Procyon B and Van Mannen's star are so high that these lead to velocities comparable with that of light and relativistic consideration are to be introduced for these. We intend to discuss these cases in a subsequent paper.

From Table II it is interesting to note that D, κ and ϑ are affected but little by the variation of temperature for the same star. The density however has a marked effect and this is the decisive factor in the degenerate state. For instance κ in contrast to the classical case is found to decrease with the number but slightly increases with temperature as in the classical case. A detailed discussion of the bearing of these on stellar structure will be given elsewhere.

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